

PPT from spectra

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Abstract

In this contribution we solve the following problem. Let H_{nm} be a Hilbert space of dimension nm , and let A be a positive semidefinite self-adjoint linear operator on H_{nm} . Under which conditions on the spectrum has A a positive partial transpose (is PPT) with respect to any partition $H_n \otimes H_m$ of the space H_{nm} as a tensor product of an n -dimensional and an m -dimensional Hilbert space? We show that the necessary and sufficient conditions can be expressed as a set of linear matrix inequalities (LMIs) on the eigenvalues of A .

1 Introduction

This paper is motivated by a problem posed by Emanuel Knill and listed as Problem 15 in the list of open problems in Quantum Information Theory on the website of the Institute of Mathematical Physics at the TU Braunschweig [1]. Knill posed the following question.

Suppose A is a self-adjoint PSD operator on a Hilbert space H_{nm} of dimension $N = nm$. Which are the conditions on the spectrum of A guaranteeing that A is separable with respect to any decomposition of H_{nm} as a tensor product $H_n \otimes H_m$ of two Hilbert spaces of dimensions n, m , respectively?

The problem was solved for the case $n = m = 2$ (a 2-qubit system) by Verstraete et al. [3], who showed that a necessary and sufficient condition is represented by a quadratic inequality on the eigenvalues of A .

It is well-known that a necessary condition for separability is the so-called *positive partial transpose* (PPT) condition [4],[2]. Therefore any condition on the spectrum of A that is necessary for the PPT property to hold with respect to any decomposition $H_{nm} = H_n \otimes H_m$ will also be necessary for separability with respect to any decomposition.

In this contribution we explicit necessary and sufficient conditions on the spectrum under which A exhibits the PPT property (is PPT) with respect to arbitrary decompositions of fixed dimension. These conditions are expressed as linear matrix inequalities (LMIs) on the eigenvalues of A . Since the PPT property is equivalent to separability for the cases $m = 2, n = 2, 3$ [4], we solve the original problem of E. Knill for these two cases and thus furnish an exact solution for a second special case. For arbitrary dimensions the obtained LMIs are necessary conditions for separability. The number of LMIs depends only on the minimum $\min(n, m)$ of the factor dimensions.

Another result that emerged from our study is that the property of being PPT for arbitrary decompositions of H_N into a tensor product of spaces of fixed dimensions n, m gets stronger if $\min(n, m)$ gets bigger. In other words, let $N = nm = n'm'$ with $\min(n, m) \geq \min(n', m')$. If a PSD operator A is PPT with respect to arbitrary decompositions $H_N = H_n \otimes H_m$, then it is also PPT with respect to arbitrary decompositions $H_N = H_{n'} \otimes H_{m'}$.

The remainder of the paper is structured as follows. In the next section we derive the necessary and sufficient conditions on the spectrum of A in the form of LMIs. In Section 3 we show the above-mentioned dependence of the stringence of the condition on the dimensions of the decomposition. In the fourth section we illustrate our results for the cases $\min(n, m) = 2, 3$ and list the corresponding LMI conditions explicitly. In the last section we summarize our results.

In the sequel H_n denotes a Hilbert space of dimension n . Let A be a hermitian matrix of size $nm \times nm$. We consider A as consisting of $m \times m$ blocks of size $n \times n$ each. The *partial transpose* of A , denoted by A^Γ , will be defined as an $nm \times nm$ -matrix consisting of the same blocks, but with the blocks (k, l) , (l, k) interchanged for $1 \leq k < l \leq m$.

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2 Main theorem

In this section we derive a set of LMIs on the spectrum of a self-adjoint PSD operator A on H_{nm} which represents a necessary and sufficient condition for A to have the PPT property with respect to any decomposition of H_{nm} as a tensor product $H_n \otimes H_m$.

Let $x \in \mathbf{R}^n$ be an arbitrary real vector of dimension n . Let us define a set $E(x)$ by

$$E(x) = \{x_k^2, (k = 1, \dots, n); +x_k x_l, -x_k x_l, (1 \leq k < l \leq n)\}.$$

This set contains n^2 real numbers.

Our main result is based on the following lemma.

Lemma 1 *Let $B = bb^*$ be a hermitian PSD rank 1 matrix of size nm . Here b is a vector in H_{nm} . Then there exists a real vector $x \in \mathbf{R}^p$ with non-negative, ordered entries $x_1 \geq x_2 \geq \dots \geq x_p \geq 0$, where $p = \min(n, m)$, such that the spectrum of the partial transpose B^Γ is given by the set $E(x)$, the remaining $p|n - m|$ eigenvalues being zero. The elements x_k , $k = 1, \dots, p$ are the singular values of the $n \times m$ matrix \mathbf{b} that is obtained from b by arranging the m n -dimensional subvectors b_1, \dots, b_m of b columnwise.*

Remark: Obviously the assertion of the lemma holds also if $B = 0$.

Proof. Let $B = bb^*$ satisfy the assumptions of the lemma, let b_1, \dots, b_m be the n -dimensional subvectors of b and let \mathbf{b} be the $n \times m$ matrix composed of these subvectors.

Let now $\mathbf{b} = U_n D V_m$ be the singular value decomposition of the matrix \mathbf{b} , where U_n, V_m are unitary operators of appropriate sizes and D a diagonal matrix of size $n \times m$ containing the singular values x_1, \dots, x_p of \mathbf{b} in decreasing order.

Then $W = U_n^* \otimes \bar{V}_m$, $W_\Gamma = U_n^* \otimes V_m$ are unitary operators on H_{nm} . It is easily seen that the relation $(WBW^*)^\Gamma = W_\Gamma B^\Gamma W_\Gamma^*$ holds. In particular, the spectrum of B^Γ equals the spectrum of $(WBW^*)^\Gamma$.

Let us determine the structure of $(WBW^*)^\Gamma$. Let v_1, \dots, v_m be the rows of V_m . It is not hard to see that

$$w = Wb = (U_n^* \otimes \bar{V}_m) \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} U_n^* \mathbf{b} v_1^* \\ \vdots \\ U_n^* \mathbf{b} v_m^* \end{pmatrix} = \text{vec}(D),$$

where the operator vec stacks the columns of the matrix it is applied to into a big column vector. Hence the vector $w = Wb$ has x_k as $(k + (k-1)n)$ -th element, $k = 1, \dots, p$, and all other elements are zero. Partition the matrix $WBW^* = ww^*$ into $m \times m$ blocks of size $n \times n$. The block at position (k, l) ($k, l \leq p$) has the product $x_k x_l$ at position (k, l) , all other elements being zero.

Let us consider the partial transpose $(WBW^*)^\Gamma = (ww^*)^\Gamma$ of this matrix. Its block at position (k, l) has the product $x_k x_l$ at position (l, k) ($k, l \leq p$), all other elements being zero. It is not hard to see that this matrix can be block-diagonalised by a permutation of the rows and columns, with diagonal blocks

$$x_k^2, (k = 1, \dots, p); \begin{pmatrix} 0 & x_k x_l \\ x_k x_l & 0 \end{pmatrix}, (1 \leq k < l \leq p),$$

all other blocks being zero. Therefore the spectrum of the matrix $(WBW^*)^\Gamma$ is given by x_k^2 , $k = 1, \dots, p$; $+x_k x_l, -x_k x_l$, $1 \leq k < l \leq p$, the rest of the eigenvalues being zero. The first p^2 numbers form exactly the set $E(x)$, where x is the vector composed of the singular values x_1, \dots, x_p of \mathbf{b} . This completes the proof of the lemma. \square

Lemma 2 *Let n, m be positive integers and let $p = \min(n, m)$. Let further $x \in \mathbf{R}^p$ be an arbitrary vector. Then there exists a vector $b \in H_{nm}$ such that the spectrum of $(bb^*)^\Gamma$ is given by the set $E(x)$, the remaining $p|n - m|$ eigenvalues being zero.*

Proof. Define b elementwise as follows. Let x_k be the $(k + (k-1)n)$ -th element of b , $k = 1, \dots, p$, and let all other elements be zero. Let the matrix \mathbf{b} , as in the previous lemma, be composed columnwise of the m n -dimensional subvectors b_1, \dots, b_m of b . Then \mathbf{b} is diagonal with x_k as diagonal elements. Hence its singular values are the absolute values $|x_k|$. Applying Lemma 1, we get that the eigenvalues of $(bb^*)^\Gamma$ are given by the products $|x_k|^2$, $k = 1, \dots, p$; $+|x_k||x_l|, -|x_k||x_l|$, $1 \leq k < l \leq p$, the rest of the eigenvalues being zero. But this is exactly the spectrum we claimed $(bb^*)^\Gamma$ to possess. \square

Lemma 3 *Let A, B be hermitian matrices of size $n \times n$. Let a_1, \dots, a_n and b_1, \dots, b_n be their eigenvalues in decreasing order. Then*

$$\min_{U \text{ unitary}} \langle UAU^*, B \rangle = \sum_{k=1}^n a_{n+1-k} b_k.$$

Proof. The scalar product $\langle UAU^*, B \rangle$ is an analytic function $f(U)$ of the unitary matrix U , which ranges over a compact set. Let U_0 realize the minimum f^* of this function. Then the first order extremality condition states that for any skew-hermitian matrix S we have $\langle SU_0AU_0^*, B \rangle + \langle U_0AU_0^*S^*, B \rangle = \langle S, [U_0AU_0^*, B] \rangle = 0$. In other words, the commutator $[U_0AU_0^*, B]$ is hermitian. Since the commutator of two hermitian matrices is always skew-hermitian, the matrices $U_0AU_0^*$ and B must commute. But then we can diagonalize them simultaneously by conjugation with some unitary matrix V . Moreover, we have $\langle U_0AU_0^*, B \rangle = \langle VU_0AU_0^*V^*, VBV^* \rangle$.

Therefore the minimum f^* is given by $\min_{\sigma \in S_n} \sum_{k=1}^n a_{\sigma(k)} b_k$, where σ ranges over all permutations of the indices $1, \dots, n$. This minimum is attained at the inversion σ^* defined by $\sigma^*(k) = n + 1 - k$. Indeed, let $\sigma \in S_n$ and $k < l$ be such that $\sigma(k) < \sigma(l)$. Then $a_{\sigma(k)} \leq a_{\sigma(l)}$ and $b_k \leq b_l$. It follows that $(a_{\sigma(k)} - a_{\sigma(l)})(b_k - b_l) \geq 0$ and hence $a_{\sigma(k)}b_k + a_{\sigma(l)}b_l \geq a_{\sigma(l)}b_k + a_{\sigma(k)}b_l$. Therefore an interchange of $\sigma(k)$ and $\sigma(l)$ can only decrease the value of the sum $\sum_{k=1}^n a_{\sigma(k)}b_k$. Performing this interchange consecutively for all pairs (k, l) for which $k < l$ and $\sigma(k) < \sigma(l)$, we finally arrive at the inversion σ^* , regardless of the permutation σ we started with. This completes the proof. \square

Let A be a self-adjoint PSD operator on H_{nm} . Let $\lambda_1, \dots, \lambda_{nm}$ be the eigenvalues of A in decreasing order and let $p = \min(n, m)$. Define also $p_+ = p(p+1)/2$, $p_- = p(p-1)/2$ and let $S_+ = \{(k, l) \mid 1 \leq k \leq l \leq p\}$, $S_- = \{(k, l) \mid 1 \leq k < l \leq p\}$ be sets of index pairs with cardinalities p_+, p_- , accordingly. Note that we have $E(x) = \{x_k x_l \mid (k, l) \in S_+\} \cup \{-x_k x_l \mid (k, l) \in S_-\}$ for $x \in \mathbf{R}^p$. In the sequel we will consider orderings of the sets S_+, S_- . We define an ordering of a finite set S as a bijective map σ from S onto the set $\{1, 2, \dots, \#S\}$, where $\#S$ is the cardinality of S .

Let $x \in \mathbf{R}^p$ be a vector with non-negative entries.

Definition 1 We say that an ordering $\sigma_+ : S_+ \rightarrow \{1, \dots, p_+\}$ of S_+ is compatible with x if for any two index pairs $(k_1, l_1), (k_2, l_2) \in S_+$ such that $\sigma_+(k_1, l_1) < \sigma_+(k_2, l_2)$ we have $x_{k_1} x_{l_1} \geq x_{k_2} x_{l_2}$.

We say that a pair of orderings $(\sigma_+ : S_+ \rightarrow \{1, \dots, p_+\}, \sigma_- : S_- \rightarrow \{1, \dots, p_-\})$ of the sets S_+, S_- is compatible with x if σ_+ is compatible with x and for any two index pairs $(k_1, l_1), (k_2, l_2) \in S_-$ such that $\sigma_+(k_1, l_1) < \sigma_+(k_2, l_2)$ (recall that $S_- \subset S_+$) we have $\sigma_-(k_1, l_1) < \sigma_-(k_2, l_2)$.

Thus for a compatible ordering 1 is the image of the index pair $(k, l) \in S_{\pm}$ for which the product $x_k x_l$ is largest, 2 the image of the pair for which $x_k x_l$ is second-largest and so on. Note that for any x there exists at least one pair of orderings which is compatible with x .

Corollary 1 The operator A is PPT for all decompositions of H_{nm} as a tensor product space $H_n \otimes H_m$ if and only if for any vector $x \in \mathbf{R}^p$ with non-negative and ordered entries $x_1 \geq x_2 \geq \dots \geq x_p \geq 0$ there exists a pair (σ_+, σ_-) of orderings which is compatible with x such that

$$\sum_{(k,l) \in S_+} \lambda_{nm+1-\sigma_+(k,l)} x_k x_l - \sum_{(k,l) \in S_-} \lambda_{\sigma_-(k,l)} x_k x_l \geq 0. \quad (1)$$

Proof. By definition, A is PPT for all decompositions of H_{nm} if for all unitary matrices U the partial transpose $(UAU^*)^\Gamma$ is PSD. This is the case if for all vectors $b \in H_{nm}$ we have

$$b^*(UAU^*)^\Gamma b = \langle (UAU^*)^\Gamma, bb^* \rangle = \langle UAU^*, (bb^*)^\Gamma \rangle \geq 0.$$

Let us fix b for the moment. By Lemma 1 there exist non-negative numbers $x_1 \geq \dots \geq x_p$ such that the spectrum of $(bb^*)^\Gamma$ consists of the set $E(x)$, the rest of the eigenvalues being zero. Note that, apart from the zeros, there are p_+ non-negative eigenvalues $x_k x_l$, $(k, l) \in S_+$, and p_- non-positive eigenvalues $-x_k x_l$, $(k, l) \in S_-$. If (σ_+, σ_-) is any pair of orderings compatible with x , then we have by Lemma 3

$$\min_U b^*(UAU^*)^\Gamma b = \min_U \langle UAU^*, (bb^*)^\Gamma \rangle = \sum_{(k,l) \in S_+} \lambda_{nm+1-\sigma_+(k,l)} x_k x_l - \sum_{(k,l) \in S_-} \lambda_{\sigma_-(k,l)} x_k x_l.$$

If the expression on the right-hand side is non-negative for any vector x with non-negative and ordered entries, then the expression on the left-hand side is non-negative for all vectors b . Hence in this case A is PPT for all decompositions of H_{nm} .

Let now $x \in \mathbf{R}^p$ be a vector with non-negative entries, and (σ_+, σ_-) a pair of orderings which is compatible with x . Suppose that

$$\sum_{(k,l) \in S_+} \lambda_{nm+1-\sigma_+(k,l)} x_k x_l - \sum_{(k,l) \in S_-} \lambda_{\sigma_-(k,l)} x_k x_l < 0.$$

By Lemmas 2 and 3 there exists a vector $b \in H_{nm}$ such that

$$\min_U b^*(UAU^*)^\Gamma b = \sum_{(k,l) \in S_+} \lambda_{nm+1-\sigma_+(k,l)} x_k x_l - \sum_{(k,l) \in S_-} \lambda_{\sigma_-(k,l)} x_k x_l < 0.$$

Let U_0 be the unitary matrix that realizes this minimum. Then we get that $U_0 A U_0^*$ is not PPT. Thus if A is PPT for all decompositions of H_{nm} , then inequality (1) holds for any vector x with non-negative entries and any pair of orderings (σ_+, σ_-) which is compatible with x . But such pairs do exist for any vector x . This completes the proof. \square

Let us transform expression (1). Let (σ_+, σ_-) be a pair of orderings. Define a $p \times p$ -matrix $\Lambda(\sigma_+, \sigma_-)$ elementwise as follows.

$$\Lambda_{kl}(\sigma_+, \sigma_-) = \begin{cases} \lambda_{nm+1-\sigma_+(k,l)}, & k \leq l, \\ -\lambda_{\sigma_-(l,k)}, & k > l. \end{cases}$$

Then we have

$$\sum_{(k,l) \in S_+} \lambda_{nm+1-\sigma_+(k,l)} x_k x_l - \sum_{(k,l) \in S_-} \lambda_{\sigma_-(k,l)} x_k x_l = x^T \Lambda(\sigma_+, \sigma_-) x. \quad (2)$$

The matrix Λ has thus signed eigenvalues of A as elements, namely the p_+ smallest eigenvalues in the upper triangular part, including the diagonal, and the p_- largest eigenvalues with a minus sign in the lower triangular part. The particular arrangement depends on the orderings σ_+, σ_- . Define the finite set of pairs

$$\Sigma_{\pm} = \{(\sigma_+, \sigma_-) \mid \exists x_1 > x_2 > \dots > x_p > 0 : (\sigma_+, \sigma_-) \text{ compatible with } x = (x_1, \dots, x_p)^T\}.$$

Lemma 4 *Let $x \in \mathbf{R}^p$ be a vector with non-negative ordered entries $x_1 \geq \dots \geq x_p \geq 0$. Then there exists a pair $(\sigma_+, \sigma_-) \in \Sigma_{\pm}$ which is compatible with x .*

Proof. Let x satisfy the assumptions of the lemma. Then there exists a sequence of vectors $x^k \in \mathbf{R}^p$ such that $\lim_{k \rightarrow \infty} x^k = x$ and the components of x^k satisfy the inequalities $x_1^k > x_2^k > \dots > x_p^k > 0$ for all k . By the definition of Σ_{\pm} , for any k there exists a pair of orderings $(\sigma_+^k, \sigma_-^k) \in \Sigma_{\pm}$ that is compatible with x^k . Let (σ_+, σ_-) be an accumulation point of the sequence $\{(\sigma_+^k, \sigma_-^k)\}$. Obviously (σ_+, σ_-) is in Σ_{\pm} and compatible with x . \square

Now we are ready to prove our main theorem.

Theorem 1 *The operator A is PPT for all decompositions of H_{nm} as a tensor product space $H_n \otimes H_m$ if and only if for all $(\sigma_+, \sigma_-) \in \Sigma_{\pm}$ we have*

$$\Lambda(\sigma_+, \sigma_-) + \Lambda(\sigma_+, \sigma_-)^T \succeq 0.$$

Proof. Suppose that $\Lambda(\sigma_+, \sigma_-) + \Lambda(\sigma_+, \sigma_-)^T \succeq 0$ for all $(\sigma_+, \sigma_-) \in \Sigma_{\pm}$. Then $x^T \Lambda(\sigma_+, \sigma_-) x \geq 0$ for all $x \in \mathbf{R}^p$ and for all $(\sigma_+, \sigma_-) \in \Sigma_{\pm}$. From Corollary 1, relation (2) and Lemma 4 it follows that A is PPT for all decompositions of H_{nm} .

Let now $(\sigma_+, \sigma_-) \in \Sigma_{\pm}$ and $x \in \mathbf{R}^p$ be such that $x^T \Lambda(\sigma_+, \sigma_-) x < 0$. We shall show that there exists a unitary matrix U such that $U A U^*$ is not PPT. By Lemma 2 there exists a vector $b \in H_{nm}$ such that the matrix $(bb^*)^\Gamma$ has the spectrum $E(x)$, the rest of the eigenvalues being zero. Denote the eigenvalues of $(bb^*)^\Gamma$ by μ_1, \dots, μ_{nm} .

The expression on the left-hand side of (2) is of the form

$$\sum_{k=1}^{nm} \lambda_{\sigma(k)} \mu_k,$$

where σ is a permutation of the index set $\{1, \dots, nm\}$. There exist unitary matrices U, V such that $U A U^* = \text{diag}(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \dots, \lambda_{\sigma(nm)})$, $V(bb^*)^\Gamma V^* = \text{diag}(\mu_1, \dots, \mu_{nm})$. Hence

$$x^T \Lambda(\sigma_+, \sigma_-) x = \langle U A U^*, V(bb^*)^\Gamma V^* \rangle = \langle (V^* U A U^* V)^\Gamma, bb^* \rangle < 0,$$

and $V^* U A U^* V$ is not PPT. This completes the proof. \square

Thus we have expressed the sought necessary and sufficient conditions on the spectrum of A as a set of LMIs.

3 Decompositions of different dimensions

In this section we prove that if $N = n_1 m_1 = n_2 m_2$ with $\min(n_1, m_1) \geq \min(n_2, m_2)$, than any self-adjoint PSD operator A on H_N which is PPT for any decomposition $H_N = H_{n_1} \otimes H_{m_1}$, is also PPT for any decomposition $H_N = H_{n_2} \otimes H_{m_2}$.

In the sequel we explicate the dependence on the dimension p of the sets S_+, S_-, Σ_{\pm} defined in the previous section, i.e. we write $S_+(p), S_-(p), \Sigma_{\pm}(p)$. Our result is based on the following lemma.

Lemma 5 Let $p > q$ and suppose that the elements of the vector $y \in \mathbf{R}^q$ satisfy the inequalities $y_1 > y_2 > \dots > y_q > 0$. Let (σ_+, σ_-) be a pair of orderings of the sets $S_+(q), S_-(q)$ that is compatible with y . Then there exists a vector $x \in \mathbf{R}^p$ with elements satisfying the inequalities $x_1 > x_2 > \dots > x_p > 0$ and a pair of orderings (ρ_+, ρ_-) of the sets $S_+(p), S_-(p)$ which is compatible with x such that ρ_+ equals σ_+ on the domain $S_+(q)$ of definition of σ_+ and ρ_- equals σ_- on the domain $S_-(q)$ of definition of σ_- .

Proof. Let y and (σ_+, σ_-) satisfy the assumptions of the lemma. Define the vector x as follows. For $k \leq q$ let $x_k = y_k$. For $k > q$ choose x_k such that $\frac{y_q^2}{y_1} > x_{q+1} > x_{q+2} > \dots > x_p > 0$.

Since $\frac{y_q}{y_1} < 1$, we have $x_q = y_q > \frac{y_q^2}{y_1} > x_{q+1}$. Therefore the sequence $\{x_k\}$ is strictly decreasing. Further, let $(k, l) \in S_+(q)$ and $(k', l') \in S_+(p) \setminus S_+(q)$. Then we have $l' > q$. It follows that $x_{k'} x_{l'} \leq x_1 x_{q+1} < x_1 \frac{y_q^2}{y_1} = x_q^2 \leq x_k x_l$. Therefore $\rho_+(k', l') > \rho_+(k, l)$ for any ordering ρ_+ that is compatible with x . As a consequence, any such ordering maps $S_+(q)$ onto the set $\{1, \dots, \#S_+(q)\}$. Let ρ_+ be an ordering of $S_+(p)$ which is compatible with x . Define another ordering ρ'_+ of $S_+(p)$ by

$$\rho'_+(k, l) = \begin{cases} \sigma_+(k, l), & (k, l) \in S_+(q), \\ \rho_+(k, l), & (k, l) \in S_+(p) \setminus S_+(q). \end{cases}$$

Since σ_+ is compatible with y , and $x_k = y_k$ for $k \leq q$, we have also that ρ'_+ is compatible with x . Now it rests to choose ρ'_- as the unique ordering of $S_-(p)$ that makes the pair (ρ'_+, ρ'_-) compatible with x . \square

Now we are ready to prove the above-mentioned result.

Theorem 2 Let $N = n_1 m_1 = n_2 m_2$ with $p_1 = \min(n_1, m_1) \geq p_2 = \min(n_2, m_2)$. Let A be a self-adjoint PSD operator on H_N which is PPT with respect to any decomposition $H_N = H_{n_1} \otimes H_{m_1}$. Then A is also PPT with respect to any decomposition $H_N = H_{n_2} \otimes H_{m_2}$.

Proof. Assume the conditions of the theorem. Let $(\sigma_+, \sigma_-) \in \Sigma_{\pm}(p_2)$. By the preceding lemma there exists a pair of orderings $(\rho_+, \rho_-) \in \Sigma_{\pm}(p_1)$ such that ρ_+ equals σ_+ on $S_+(p_2)$ and ρ_- equals σ_- on $S_-(p_2)$. It follows that the upper left $p_2 \times p_2$ -subblock of the matrix $\Lambda(\rho_+, \rho_-)$ equals the matrix $\Lambda(\sigma_+, \sigma_-)$. But $\Lambda(\rho_+, \rho_-) + \Lambda(\rho_+, \rho_-)^T \succeq 0$ by Theorem 1 and the assumption on A . Hence we have also $\Lambda(\sigma_+, \sigma_-) + \Lambda(\sigma_+, \sigma_-)^T \succeq 0$. Application of Theorem 1 completes the proof. \square

4 Examples

In this section we illustrate the obtained results on the examples of $2 \times n$ and $3 \times n$ bipartite spaces.

Let $m = 2, n \geq 2$. Then we have $p = 2, p_+ = 3, p_- = 1$. Let us find the set $\Sigma_{\pm}(2)$. If $x \in \mathbf{R}^2$ is a vector with elements $x_1 > x_2 > 0$, then we have the relations $x_1^2 > x_1 x_2 > x_2^2$ on the products $x_k x_l, (k, l) \in S_+(2)$. Hence the only element of $\Sigma_{\pm}(2)$ is the pair (σ_+, σ_-) defined by

$$\sigma_+ : \begin{cases} (1, 1) \mapsto 1, \\ (1, 2) \mapsto 2, \\ (2, 2) \mapsto 3, \end{cases} \quad \sigma_- : (1, 2) \mapsto 1.$$

The corresponding matrix Λ is given by

$$\Lambda(\sigma_+, \sigma_-) = \begin{pmatrix} \lambda_{2n} & \lambda_{2n-1} \\ -\lambda_1 & \lambda_{2n-2} \end{pmatrix}.$$

Hence we obtain the following necessary and sufficient LMI condition:

$$\begin{pmatrix} 2\lambda_{2n} & \lambda_{2n-1} - \lambda_1 \\ \lambda_{2n-1} - \lambda_1 & 2\lambda_{2n-2} \end{pmatrix} \succeq 0.$$

Since $\lambda_k \geq 0$ by the positivity of A , and $\lambda_1 \geq \lambda_{2n-1}$, this matrix inequality reduces to

$$4\lambda_{2n}\lambda_{2n-2} - (\lambda_{2n-1} - \lambda_1)^2 \geq 0 \Leftrightarrow \lambda_1 \leq \lambda_{2n-1} + 2\sqrt{\lambda_{2n}\lambda_{2n-2}}.$$

We obtain the following corollary.

Corollary 2 Let A be a self-adjoint PSD operator on the Hilbert space H_{2n} . Let $\lambda_1, \dots, \lambda_{2n}$ be the eigenvalues of A in decreasing order. Then A is PPT with respect to any decomposition of H_{2n} as a tensor product $H_2 \otimes H_n$ if and only if the inequality $\lambda_1 \leq \lambda_{2n-1} + 2\sqrt{\lambda_{2n}\lambda_{2n-2}}$ holds. \square

For $n = 2$ and $n = 3$ this inequality takes the forms

$$\lambda_1 \leq \lambda_3 + 2\sqrt{\lambda_2\lambda_4}, \quad (3)$$

$$\lambda_1 \leq \lambda_5 + 2\sqrt{\lambda_4\lambda_6}. \quad (4)$$

It is well-known [4] that for the cases $m = 2, n = 2, 3$ the PPT condition is equivalent to separability. Therefore we obtain also the following corollary.

Corollary 3 *Let A be a self-adjoint PSD operator on the Hilbert space H_4 (H_6). Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ ($\lambda_1, \dots, \lambda_6$) be the eigenvalues of A in decreasing order. Then A is separable with respect to any decomposition of H_4 (H_6) as a tensor product $H_2 \otimes H_2$ ($H_2 \otimes H_3$) if and only if inequality (3) ((4)) holds. \square*

Inequality (3) is already well-known to be a necessary and sufficient condition for separability for any partition of H_4 as $H_2 \otimes H_2$ and was presented as partial solution of Knill's *Problem 15* in quantum information theory [3]. We present here inequality (4) as partial solution for the case $m = 2, n = 3$.

Let now $m = 3, n \geq 3$. Then $p = 3, p_+ = 6, p_- = 3$. We shall now determine the set $\Sigma_{\pm}(3)$. If $x_1 > x_2 > x_3 > 0$, then we have

$$x_1^2 > x_1x_2 > \max(x_2^2, x_1x_3) \geq \min(x_2^2, x_1x_3) > x_2x_3 > x_3^2.$$

However, we can have both $x_2^2 \geq x_1x_3$ and $x_1x_3 \geq x_2^2$. Hence $\Sigma_{\pm}(3)$ consists of two elements, and the corresponding matrices Λ are given by

$$\Lambda_1 = \begin{pmatrix} \lambda_{3n} & \lambda_{3n-1} & \lambda_{3n-3} \\ -\lambda_1 & \lambda_{3n-2} & \lambda_{3n-4} \\ -\lambda_2 & -\lambda_3 & \lambda_{3n-5} \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \lambda_{3n} & \lambda_{3n-1} & \lambda_{3n-2} \\ -\lambda_1 & \lambda_{3n-3} & \lambda_{3n-4} \\ -\lambda_2 & -\lambda_3 & \lambda_{3n-5} \end{pmatrix}.$$

Thus we obtain the two LMIs

$$\begin{pmatrix} 2\lambda_{3n} & \lambda_{3n-1} - \lambda_1 & \lambda_{3n-3} - \lambda_2 \\ \lambda_{3n-1} - \lambda_1 & 2\lambda_{3n-2} & \lambda_{3n-4} - \lambda_3 \\ \lambda_{3n-3} - \lambda_2 & \lambda_{3n-4} - \lambda_3 & 2\lambda_{3n-5} \end{pmatrix} \succeq 0, \quad \begin{pmatrix} 2\lambda_{3n} & \lambda_{3n-1} - \lambda_1 & \lambda_{3n-2} - \lambda_2 \\ \lambda_{3n-1} - \lambda_1 & 2\lambda_{3n-3} & \lambda_{3n-4} - \lambda_3 \\ \lambda_{3n-2} - \lambda_2 & \lambda_{3n-4} - \lambda_3 & 2\lambda_{3n-5} \end{pmatrix} \succeq 0. \quad (5)$$

Corollary 4 *Let A be a self-adjoint PSD operator on the Hilbert space H_{3n} . Let $\lambda_1, \dots, \lambda_{3n}$ be the eigenvalues of A in decreasing order. Then A is PPT with respect to any decomposition of H_{3n} as a tensor product $H_3 \otimes H_n$ if and only if linear matrix inequalities (5) hold. \square*

5 Summary

In this contribution we presented necessary and sufficient conditions on the spectrum of a self-adjoint positive semidefinite operator A on a Hilbert space H_{nm} of dimension nm under which A has a positive partial transpose for any decomposition of H_{nm} as a tensor product space $H_n \otimes H_m$, where H_n, H_m are Hilbert spaces of dimensions n, m . These conditions have the form of linear matrix inequalities on the eigenvalues of A (Theorem 1). We showed that if these conditions hold for a pair of dimensions (n, m) , then they hold also for a pair (n', m') (where $nm = n'm'$) whenever $\min(n', m') \leq \min(n, m)$ (Theorem 2). Hence the condition is most stringent if $n \approx m$.

For the case $\min(n, m) = 2$ we reduced the LMI condition to a single inequality (Corollary 2). For the cases $m = 2, n = 2, 3$ our conditions are necessary and sufficient also for *separability* of A with respect to an arbitrary decomposition of the underlying space H_4 or H_6 (Corollary 3). While the result for $n = m = 2$ was known before, the result for $m = 2, n = 3$ is new and provides a solution of E. Knill's open problem number 15 [1] for another special case.

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